

ON THE TOPOLOGY OF ARRANGEMENTS OF A CUBIC AND ITS INFLECTIONAL TANGENTS

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ABSTRACT. A k -Artal arrangement is a reducible algebraic curve composed of a smooth cubic and k inflectional tangents. By studying the topological properties of their subarrangements, we prove that for $k = 3, 4, 5, 6$, there exist Zariski pairs of k -Artal arrangements. These Zariski pairs can be distinguished in a geometric way by the number of collinear triples in the set of singular points contained in the cubic.

1. INTRODUCTION

In this article, we continue to study Zariski pairs for reducible plane curves based on the idea used in [3]. A pair $(\mathcal{B}^1, \mathcal{B}^2)$ of reduced plane curves in \mathbb{P}^2 is said to be a Zariski pair if (i) both \mathcal{B}^1 and \mathcal{B}^2 have the same combinatorics and (ii) $(\mathbb{P}^2, \mathcal{B}^1)$ is *not* homeomorphic to $(\mathbb{P}^2, \mathcal{B}^2)$ (see [2] for details about Zariski pairs). As we have seen in [2], the study of Zariski pairs, roughly speaking, consists of two steps:

- (i) How to construct (or find) plane curves with the same combinatorics but having *some* different properties.
- (ii) How to distinguish the topology of $(\mathbb{P}^2, \mathcal{B}^1)$ and $(\mathbb{P}^2, \mathcal{B}^2)$.

As for the second step, various tools such as fundamental groups, Alexander invariants, braid monodromies, existence/non-existence of Galois covers and so on have been used. In [3], the first and last authors considered another elementary method in order to study Zariski k -plets for arrangements of reduced plane curves and showed its effectiveness by giving some new examples. In this article, we study the topology of arrangements of a smooth cubic and its inflectional tangents along the same line.

1.1. Subarrangements. We here reformulate our idea in [3] more precisely. Let \mathcal{B}_o be a (possibly empty) reduced plane curve \mathcal{B}_o . We define $\underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o}$ to be the set of the reduced plane curves of the form $\mathcal{B}_o + \mathcal{B}$, where \mathcal{B} is a reduced curve with no common component with \mathcal{B}_o .

Let $\mathcal{B} = \mathcal{B}_1 + \cdots + \mathcal{B}_r$ denote the irreducible decomposition of \mathcal{B} . For a subset \mathcal{I} of the power set $2^{\{1, \dots, r\}}$ of $\{1, \dots, r\}$, which does not contain the empty set \emptyset , we define the sub set $\underline{\text{Sub}}_{\mathcal{I}}(\mathcal{B}_o, \mathcal{B})$ of $\underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o}$ by:

$$\underline{\text{Sub}}_{\mathcal{I}}(\mathcal{B}_o, \mathcal{B}) := \left\{ \mathcal{B}_o + \sum_{i \in I} \mathcal{B}_i \mid I \in \mathcal{I} \right\}.$$

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For $\mathcal{I} = 2^{\{1, \dots, r\}} \setminus \emptyset$, we denote $\underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B}) = \underline{\text{Sub}}_{\mathcal{I}}(\mathcal{B}_o, \mathcal{B})$.

Let A be a set and suppose that a map

$$\Phi_{\mathcal{B}_o} : \underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o} \rightarrow A$$

with the following property is given: for $\mathcal{B}_o + \mathcal{B}^1, \mathcal{B}_o + \mathcal{B}^2 \in \underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o}$, if there exists a homeomorphism $h : (\mathbb{P}^2, \mathcal{B}_o + \mathcal{B}^1) \rightarrow (\mathbb{P}^2, \mathcal{B}_o + \mathcal{B}^2)$ with $h(\mathcal{B}_o) = \mathcal{B}_o$, then $\Phi_{\mathcal{B}_o}(\mathcal{B}_o + \mathcal{B}^1) = \Phi_{\mathcal{B}_o}(\mathcal{B}_o + \mathcal{B}^2)$.

We denote by $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}$ the restriction of $\Phi_{\mathcal{B}_o}$ to $\underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B})$. Note that if there exists a homeomorphism $h : (\mathbb{P}^2, \mathcal{B}_o + \mathcal{B}^1) \rightarrow (\mathbb{P}^2, \mathcal{B}_o + \mathcal{B}^2)$ for $\mathcal{B}_o + \mathcal{B}^1, \mathcal{B}_o + \mathcal{B}^2 \in \underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o}$ with $h(\mathcal{B}_o) = \mathcal{B}_o$, then we have the induced map $h_{\natural} : \underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B}^1) \rightarrow \underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B}^2)$ such that $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^1} = \tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^2} \circ h_{\natural}$:

$$\begin{array}{ccc} \underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B}^1) & & \\ \downarrow h_{\natural} & \searrow \tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^1} & \\ \underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B}^2) & \xrightarrow{\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^2}} & A \end{array}$$

Remark 1.1. In § 2 we give four explicit examples for $\Phi_{\mathcal{B}_o}$ and $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}$ allowing to distinguish the k -Artal arrangements (see § 1.2 for the definition), using the Alexander polynomial, the existence of D_6 -covers, the splitting numbers and the linking set.

If $\mathcal{D}_o + \mathcal{D}^1, \mathcal{D}_o + \mathcal{D}^2 \in \underline{\text{Curve}}_{\text{red}}^{\mathcal{D}_o}$ have same the same combinatorics, then any homeomorphism $h : (\mathcal{T}^1, \mathcal{D}_o + \mathcal{D}^1) \rightarrow (\mathcal{T}^2, \mathcal{D}_o + \mathcal{D}^2)$ induces a map $h_{\natural} : \underline{\text{Sub}}(\mathcal{D}_o, \mathcal{D}^1) \rightarrow \underline{\text{Sub}}(\mathcal{D}_o, \mathcal{D}^2)$, where \mathcal{T}^i is a tubular neighborhood of $\mathcal{D}_o + \mathcal{D}^i$ for $i = 1, 2$. Let $(\mathcal{D}_o + \mathcal{D}^1, \mathcal{D}_o + \mathcal{D}^2)$ be a Zariski pair of curves in $\underline{\text{Curve}}_{\text{red}}^{\mathcal{D}_o}$ such that

- it is distinguished by $\Phi_{\mathcal{D}_o}$, i.e., any homeomorphism $h : (\mathcal{T}^1, \mathcal{D}_o + \mathcal{D}^1) \rightarrow (\mathcal{T}^2, \mathcal{D}_o + \mathcal{D}^2)$ necessarily satisfies $h(\mathcal{D}_o) = \mathcal{D}_o$ and $\Phi_{\mathcal{D}_o}(\mathcal{D}_o + \mathcal{D}^1) \neq \Phi_{\mathcal{D}_o}(\mathcal{D}_o + \mathcal{D}^2)$, and
- the combinatorial type of $\mathcal{D}_o + \mathcal{D}^1$ and $\mathcal{D}_o + \mathcal{D}^2$ is $\underline{\mathbb{C}}$.

Assuming the existence of such a Zariski pair for the combinatorial type $\underline{\mathbb{C}}$, we construct Zariski pair with glued combinatorial type. We first note that the following proposition is immediate:

Proposition 1.2. *Choose $\mathcal{B}_o + \mathcal{B}^1, \mathcal{B}_o + \mathcal{B}^2 \in \underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o}$ with same combinatorial type. Let $\underline{\text{Sub}}_{\underline{\mathbb{C}}}(\mathcal{B}_o, \mathcal{B}^j)$ ($j = 1, 2$) be the sets of subarrangements of $\mathcal{B}_o + \mathcal{B}^j$ having the combinatorial type $\underline{\mathbb{C}}$ ($j = 1, 2$), respectively. If*

- any homeomorphism $h : (\mathcal{T}^1, \mathcal{B}_o + \mathcal{B}^1) \rightarrow (\mathcal{T}^2, \mathcal{B}_o + \mathcal{B}^2)$ necessarily satisfies $h(\mathcal{B}_o) = \mathcal{B}_o$, where \mathcal{T}^i is a tubular neighborhood of $\mathcal{B}_o + \mathcal{B}^i$ for $i = 1, 2$, and*
- for some element $a_1 \in A$,*

$$\#(\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^1}^{-1}(a_1) \cap \underline{\text{Sub}}_{\underline{\mathbb{C}}}(\mathcal{B}_o, \mathcal{B}^1)) \neq \#(\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}^2}^{-1}(a_1) \cap \underline{\text{Sub}}_{\underline{\mathbb{C}}}(\mathcal{B}_o, \mathcal{B}^2)),$$

then $(\mathcal{B}_o + \mathcal{B}^1, \mathcal{B}_o + \mathcal{B}^2)$ is a Zariski pair.

Remark 1.3. If for all automorphism σ of the combinatorics of $\mathcal{B}_o + \mathcal{B}^j$, $\sigma(\mathcal{B}_o) = \mathcal{B}_o$ then hypothesis (i) of Proposition 1.2 is always verified. In particular, it is the case if $\deg(\mathcal{B}_o) \neq \deg(\mathcal{B}_i)$, for $i = 1, \dots, r$.

1.2. Artal arrangements. In this article, we apply Proposition 1.2 to distinguish Zariski pairs formed by *Artal arrangements*. These curves are defined as follows:

Let E be a smooth cubic, let P_i ($1 \leq i \leq 9$) be its 9 inflection points and let L_{P_i} be the tangent lines at P_i ($1 \leq i \leq 9$), respectively.

Definition 1.4. Choose a subset $I \subset \{1, \dots, 9\}$. We call an arrangement $\mathcal{C} = E + \sum_{i \in I} L_{P_i}$ an Artal arrangement for I . In particular, if $k = \sharp(I)$, we call \mathcal{C} a k -Artal arrangement.

The idea is to apply Proposition 1.2 to the case when $\mathcal{B}_o = E$ and $\mathcal{B} = \sum_{i \in I} L_{P_i}$. Let \mathcal{C}^1 and \mathcal{C}^2 be two k -Artal arrangements. Note that if there exists homeomorphism $h : (\mathbb{P}^2, \mathcal{C}^1) \rightarrow (\mathbb{P}^2, \mathcal{C}^2)$, $h(E) = E$ always holds. In [1], E. Artal Bartolo gave an example of a Zariski pair for 3-Artal arrangements. Based on this example, we make use of our method to find other examples of Zariski pairs of k -Artal arrangement and obtain the following:

Theorem 1.5. *There exists Zariski pairs for k -Artal arrangement for $k = 4, 5, 6$.*

Remark 1.6. Note that the case of $k = 5$ is considered in [6]. In [6], it is shown that there exists an Zariski pair for 5-Artal arrangement.

2. SOME EXPLICIT EXAMPLES FOR $\Phi_{\mathcal{B}_o}$

We here introduce four examples for $\Phi_{\mathcal{B}_o}$. The last two were recently considered by the second author, Meilhan [6] and the third author [4], respectively.

2.1. D_{2p} -covers. For terminologies and notation, we use those introduced in [2], §3 freely.

Let D_{2p} be the dihedral group of order $2p$. Let $\text{Cov}_b(\mathbb{P}^2, 2\mathcal{B}, D_{2p})$ be the set of isomorphism classes of D_{2p} -covers branched at $2\mathcal{B}$.

We now define $\Phi_{\mathcal{B}_o}^{D_{2p}} : \underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o} \rightarrow \{0, 1\}$ as follows:

$$\Phi_{\mathcal{B}_o}^{D_{2p}}(\mathcal{B}_o + \mathcal{B}) = \begin{cases} 1 & \text{if } \text{Cov}_b(\mathbb{P}^2, 2(\mathcal{B}_o + \mathcal{B}), D_{2p}) \neq \emptyset \\ 0 & \text{if } \text{Cov}_b(\mathbb{P}^2, 2(\mathcal{B}_o + \mathcal{B}), D_{2p}) = \emptyset \end{cases}$$

Note that $\Phi_{\mathcal{B}_o}^{D_{2p}}$ satisfies the required condition described in the Introduction. Thus, we define the map $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}^{D_6}$ as the restriction of $\Phi_{\mathcal{B}_o}^{D_6}$ to $\underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B})$.

2.2. Alexander polynomials. For the Alexander polynomials of reduced plane curves, see [2], § 2. Let $\Delta : \underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o} \rightarrow \mathbb{C}[t]$ be the map assigning to a curve of $\underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o}$ its Alexander polynomial. We define the map $\Phi_{\mathcal{B}_o}^{\text{Alex}} : \underline{\text{Curve}}_{\text{red}}^{\mathcal{B}_o} \rightarrow \{0, 1\}$ by:

$$\Phi_{\mathcal{B}_o}^{\text{Alex}}(\mathcal{B}_o + \mathcal{B}') = \begin{cases} 1 & \text{if } \Delta(\mathcal{B}_o + \mathcal{B}') \neq 1 \\ 0 & \text{if } \Delta(\mathcal{B}_o + \mathcal{B}') = 1. \end{cases}$$

As previously, we define $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}^{\text{Alex}}$ as the restriction of $\Phi_{\mathcal{B}_o}^{\text{Alex}}$ to $\underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B})$.

2.3. Splitting numbers. Let $\mathcal{B}_o + \mathcal{B}$ be a reduced curves such that \mathcal{B}_o is smooth. Let $\pi_m : X \rightarrow \mathbb{P}^2$ be the unique cover branched over \mathcal{B} , corresponding to the surjection of $\pi_1(\mathbb{P}^2 \setminus \mathcal{B}) \rightarrow \mathbb{Z}/m\mathbb{Z}$ sending all meridians of the \mathcal{B}_i to 1. The *splitting number* of \mathcal{B}_o for π_m , denoted by $s_{\pi_m}(\mathcal{B})$ is the number of irreducible component of the pull-back $\pi_m^* \mathcal{B}_o$ of \mathcal{B}_o by π_m (see [4] for the general definition). By [4, Proposition 1.3], the application:

$$\Phi_{\mathcal{B}_o}^{\text{split}} : \begin{cases} \text{Curve}_{\text{red}}^{\mathcal{B}_o} & \longrightarrow \mathbb{N}^* \\ \mathcal{B}_o + \mathcal{B} & \longmapsto s_{\pi_3}(\mathcal{B}_o) \end{cases} ,$$

verify the condition of Proposition 1.2. We can then define the map $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}^{\text{split}} : \underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B}) \rightarrow \mathbb{N}^*$ as the restriction of $\Phi_{\mathcal{B}_o}^{\text{split}}$ to $\underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B})$.

2.4. Linking set. Let \mathcal{B}_o be a non-empty curve, with smooth irreducible components. A cycle of \mathcal{B}_o is an S^1 embedded in \mathcal{B}_o . For $\mathcal{B}_o + \mathcal{B} \in \text{Curve}_{\text{red}}^{\mathcal{B}_o}$, we define the *linking set* of \mathcal{B}_o , denoted by $\text{lks}_{\mathcal{B}}(\mathcal{B}_o)$, as the set of classes in $H_1(\mathbb{P}^2 \setminus \mathcal{B})/\text{Ind}_{\mathcal{B}_o}$ of the cycles of \mathcal{B}_o which not intersect \mathcal{B} , where $\text{Ind}_{\mathcal{B}_o}$ is the subgroup of $H_1(\mathbb{P}^2 \setminus \mathcal{B})$ generated by the meridians in \mathcal{B}_o around the points of $\mathcal{B}_o \cap \mathcal{B}$. This definition is weaker than [6, Definition 3.9]. By [6, Theorem 3.13], the map defined by:

$$\Phi_{\mathcal{B}_o}^{\text{lks}} : \begin{cases} \text{Curve}_{\text{red}}^{\mathcal{B}_o} & \longrightarrow \mathbb{N}^* \cup \{\infty\} \\ \mathcal{B}_o + \mathcal{B} & \longmapsto \#\text{lks}_{\mathcal{B}}(\mathcal{B}_o) \end{cases}$$

verify the condition of Proposition 1.2. We can thus define the map $\tilde{\Phi}_{\mathcal{B}_o, \mathcal{B}}^{\text{lks}}$ as the restriction of $\Phi_{\mathcal{B}_o}^{\text{lks}}$ to $\underline{\text{Sub}}(\mathcal{B}_o, \mathcal{B})$.

3. THE GEOMETRY OF INFLECTION POINTS OF A SMOOTH CUBIC

Let $E \subset \mathbb{P}^2$ be a smooth cubic curve and let $O \in E$ be an inflection point of E . In this section we consider the elliptic curve (E, O) . The following facts are well known :

- (1) The set of inflection points of E can be identified with $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \subset E$, the subgroup of three torsion points of E .
- (2) Let P, Q, R be distinct inflection points of E . Then P, Q, R are collinear if and only if $P + Q + R = O \in (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$.

From the above facts we can study the geometry of inflection points and the following proposition follows:

Proposition 3.1. *Let E be a cubic curve and $\{P_1, \dots, P_k\} \in E$ be a set of distinct inflection points of E . Let n be the number of triples $\{P_{i_1}, P_{i_2}, P_{i_3}\} \subset \{P_1, \dots, P_k\}$ such that they are collinear. Then the possible values of n for $k = 3, \dots, 9$ are as in the following table:*

k	3	4	5	6	7	8	9
n	0, 1	0, 1	1, 2	2, 3	5	8	12

4. PROOF OF THE MAIN THEOREM

4.1. The case of 3-Artal arrangements. Using the four invariants introduced in § 2, we can prove the original result of E. Artal. Let $I = \{i_1, i_2, i_3\} \subset \{1, \dots, 9\}$ and $\mathcal{L}_I = \sum_{i \in I} L_{P_i}$.

Theorem 4.1. *For a 3-Artal arrangement $\mathcal{C} = E + \sum_{i \in I} L_{P_i} = E + \mathcal{L}_I$, we have:*

$$\begin{aligned} (1) \quad \tilde{\Phi}_{E, \mathcal{L}_I}^{D_6}(\mathcal{C}) &= \begin{cases} 1 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are collinear} \\ 0 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are not collinear} \end{cases} \\ (2) \quad \tilde{\Phi}_{E, \mathcal{L}_I}^{\text{Alex}}(\mathcal{C}) &= \begin{cases} 1 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are collinear} \\ 0 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are not collinear} \end{cases} \\ (3) \quad \tilde{\Phi}_{E, \mathcal{L}_I}^{\text{split}}(\mathcal{C}) &= \begin{cases} 3 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are collinear} \\ 1 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are not collinear} \end{cases} \\ (4) \quad \tilde{\Phi}_{E, \mathcal{L}_I}^{\text{lks}}(\mathcal{C}) &= \begin{cases} 1 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are collinear} \\ 3 & \text{if } P_{i_1}, P_{i_2}, P_{i_3} \text{ are not collinear} \end{cases} \end{aligned}$$

Proof. (1) This is the result of the last author ([5]).

(2) This is the result of E. Artal ([1]).

(3) By [4, Theorem 2.7], we obtain $\Phi^{\text{split}}(\mathcal{C}, E) = 3$ if the three tangent points are collinear, and $\Phi^{\text{split}}(\mathcal{C}, E) = 1$ otherwise.

(4) Using the same arguments as in [7], we can prove that, in the case of 3-Artal arrangement, $s_{\mathcal{B}_o}(\mathcal{C}) = \frac{3}{\# \text{lks}_{\mathcal{B}}(\mathcal{B}_o)}$. Using the previous point we obtain the result. \square

Remark 4.2. It is also possible to consider $\tilde{\Phi}_{\mathcal{L}_J, E}^{\text{lks}}(E)$. But in this case, we have no method to compute it in the general case. But, if C is the cubic defined by $x^3 - xz^2 - y^2z = 0$, the computation done in [6] implies the result.

Corollary 4.3. *Choose $\{i_1, i_2, i_3, i_4\} \subset \{1, \dots, 9\}$ such that $P_{i_1}, P_{i_2}, P_{i_3}$ are collinear, while P_{i_1}, P_{i_2} and P_{i_4} are not collinear. Put $\mathcal{C}_1 = E + L_{i_1} + L_{i_2} + L_{i_3}$ and $\mathcal{C}_2 = E + L_{i_1} + L_{i_2} + L_{i_4}$. Then $(\mathcal{C}_1, \mathcal{C}_2)$ is a Zariski pair.*

4.2. The other cases. Choose a subset J of $\subset \{1, \dots, 9\}$ such that $4 \leq \#J \leq 6$ and let

$$\mathcal{C} := E + \mathcal{L}_J, \quad \mathcal{L}_J = \sum_{j \in J} L_{P_j},$$

be a k -Artal arrangement. To distinguish these arrangements in a geometric way (as the collinearity in the case of 3-Artal arrangement), let us introduce the type of a k -Artal arrangement.

Definition 4.4. *For $k = 4, 5, 6$, we say an arrangement of the form $\mathcal{C} = E + L_{P_1} + \dots + L_{P_k}$ to be of Type I if the number n of collinear triples in $\{P_1, \dots, P_k\}$ is $n = k - 3$, while we say \mathcal{C} to be of Type II if the number n of collinear triples in $\{P_1, \dots, P_k\}$ is $n = k - 4$.*

Theorem 4.5. *Let \mathcal{C}_1 be an arrangement of Type I and \mathcal{C}_2 be an arrangement of Type II. Then $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ are not homeomorphic as pairs.*

Furthermore if \mathcal{C}_1 and \mathcal{C}_2 have the same combinatorics, $\mathcal{C}_1, \mathcal{C}_2$ form a Zariski pair.

Proof. Let \mathcal{C} be a k -Artal arrangement ($k = 4, 5, 6$). We denote by $\underline{\text{Sub}}(E, \mathcal{L}_J)_3$ the set of 3-Artal arrangements contained in $\underline{\text{Sub}}(E, \mathcal{L}_J)$. Let $\Phi_{\mathcal{C},3}^{D_6}$, $\Phi_{\mathcal{C},3}^{\text{Alex}}$, $\Phi_{\mathcal{C},3}^{\text{split}}$ and $\Phi_{\mathcal{C},3}^{\text{lks}}$ be the restrictions of $\tilde{\Phi}_{E,\mathcal{L}_J}^{D_6}$, $\tilde{\Phi}_{E,\mathcal{L}_J}^{\text{Alex}}$, $\tilde{\Phi}_{E,\mathcal{L}_J}^{\text{split}}$ and $\tilde{\Phi}_{E,\mathcal{L}_J}^{\text{lks}}$ to $\underline{\text{Sub}}(E, \mathcal{L}_J)_3$, respectively. Then by Theorem 4.1, we have

$$\#(\Phi_{\mathcal{C},3}^{D_6}{}^{-1}(1)) = \#(\Phi_{\mathcal{C},3}^{\text{Alex}}{}^{-1}(1)) = \#(\Phi_{\mathcal{C},3}^{\text{split}}{}^{-1}(3)) = \#(\Phi_{\mathcal{C},3}^{\text{lks}}{}^{-1}(1)) = \begin{cases} k-3 & \text{if } \mathcal{C} \text{ is type I} \\ k-4 & \text{if } \mathcal{C} \text{ is type II.} \end{cases}$$

If a homeomorphism $h : (\mathbb{P}^2, \mathcal{C}_1) \rightarrow (\mathbb{P}^2, \mathcal{C}_2)$ exists, it follows that $h_*(\underline{\text{Sub}}(\mathcal{C}_1)_3) = \underline{\text{Sub}}(\mathcal{C}_2)_3$. This contradicts the above values. Hence our statements follow. \square

Remark 4.6. As a final remark, we note that for $k = 1, 2, 7, 8, 9$ it can be proved that there do not exist Zariski pairs consisting of k -Artal arrangements.

REFERENCES

- [1] E. Artal Bartolo: *Sur les couples de Zariski*, J. Algebraic Geom., **3**(1994), 563-597.
- [2] E. Artal Bartolo, J.-I. Cogolludo and H. Tokunaga: *A survey on Zariski pairs*, Adv.Stud.Pure Math., **50**(2008), 1-100.
- [3] S. Bannai and H. Tokunaga: *Geometry of bisections of elliptic surfaces and Zariski N -plets for conic arrangements*, Geom Dedicata, **178** (2015), 219-237, DOI 10.1007/s10711-015-0054-z.
- [4] T. Shirane: *A note on splitting numbers for Galois covers and π_1 -equivalent Zariski k -plets*, Available at [arXiv:1601.03792](https://arxiv.org/abs/1601.03792), to appear in Proc. Amer. Math. Soc.
- [5] H. Tokunaga: *A remark on Artal's paper*, Kodai Math. J.**19** (1996), 207-217.
- [6] B. Guerville-Ballé and J.-B. Meilhan: *A linking invariant for algebraic curves*, Available at [arXiv:1602.04916](https://arxiv.org/abs/1602.04916).
- [7] B. Guerville-Ballé and T. Shirane: *Equivalence between splitting number and linking invariant*, Available at [arXiv:1607.04951](https://arxiv.org/abs/1607.04951).

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